Updated analytical expressions for critical bandwidth and critical-band rate

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Introduction

The concept of critical bands as introduced based on studies of Fletcher and Munson (1937) by Zwicker et al. (1957) describes frequency bands of frequency dependent spectral width with no fixed position on the frequency scale, as they appear in various psychoacoustic experiments (cf. Fastl and Zwicker 2007, pp. 150–158). Among those are studies on loudness summation (Zwicker et al. 1957), absolute thresholds (Gässler 1954), and masking patterns of narrow-band noises (Fastl and Schorer 1986). All these experiments show a change of results if the spectral width of one of the stimuli involved is increased beyond the critical bandwidth (CBW) $\Delta f_{\mathrm{G}}(f)$ centered at the respective frequency f. Based on the critical bands, a critical-band rate (CBR) function $z(f)$ has been proposed (Zwicker 1961), relating frequency to CBR so that all critical bands are equally wide on the CBR scale. Different formulae to calculate CBR, its inverse $f(z)$, and CBW were proposed (Zwicker and Terhardt 1980, Traunmüller 1990). A similar concept based on equivalent-rectangular bandwidths (ERBs) $\Delta f_{\mathrm{E}}(f)$ (Patterson 1976) was introduced by Moore and Glasberg (1983, 1987).

The critical-band concept is frequently applied, for example in auditory-adapted Fourier transform (Terhardt 1985), speech coding (Mummert 1997), signal analysis (Völk et al. 2009, 2011), and signal processing for auditory prostheses such as hearing aids and cochlear implants. Previously introduced CBW and ERB functions specify the bandwidth symmetrically around a center frequency; however, the low-frequency bandwidths exceed double the center frequency. This confines the applicability and universality of the concept, since a band-pass filter bandwidth $\Delta f(f) > 2f$ centered at f will introduce artifacts due to selection of negative frequencies, even if ideal filtering is assumed, and can hardly be justified by psychoacoustic experiments. In fact, the formulae result in insufficient low-frequency selectivity for certain applications (Mummert 1997, pp. 9–12). The analytic expression for the frequency dependence of CBR proposed by Zwicker and Terhardt (1980) is based on values tabulated for $0 < f < 15.5\,\mathrm{kHz}$. At higher frequencies, the formula underestimates CBR, as shown below. In addition, the function is not invertible in closed form, preventing the calculation of frequencies corresponding to given CBRs, which is crucial for defining on the CBR scale equally distributed frequencies.

Aim & Requirements

In this paper, after an overview of earlier approaches, generalized analytic expressions for CBW and ERB are proposed for $0 \leq f \leq 20\,\mathrm{kHz}$, on the basis of the well-established formulae, but defined by continuous functions with $\Delta f(f) \leq 2f\ \forall f$. Further, an invertible CBR function valid for the full audible frequency range is proposed. Both, the critical bandwidth and the critical-band rate functions fit the measurement results more accurately than the original formulae and are especially intended for direct parameterization of auditory-adapted signal processing routines based on the critical-band concept.

Critical Bandwidth: Concept & Formulae

The CBW was estimated level independently as a function of frequency based on psychoacoustic measurements with different methods on more than 50 subjects (Fastl and Zwicker 2007, p. 185). Originally, the CBWs $\Delta f_{\mathrm{G}}[n]$ were tabulated by Zwicker (1961) as sample points $f_{\mathrm{c}}[n]$ with $n = 1, \ldots, 24$, indicated by the black dots in figure 1.

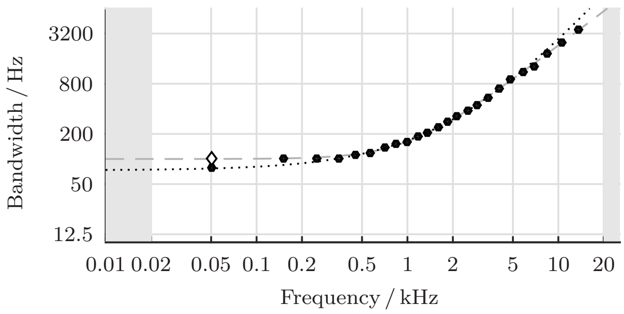

Figure 1: Critical bandwidth $\Delta f_{\mathrm{G}}(f)$ as a function of frequency f ac. to Zwicker and Terhardt (1980, dashed gray contour) and Traunmüller (1990, dotted black contour). Black dots: originally tabulated values (Zwicker 1961); unfilled diamond: update ac. to Zwicker and Terhardt (1980); light gray bars indicate the limits of the audible frequency range.

The tabulated values were reprinted by Zwicker and Terhardt (1980) with a modification: the lowest CBW $\Delta f_{\mathrm{G}}[1] = 80\,\mathrm{Hz}$ was changed to $100\,\mathrm{Hz}$ (diamond in figure 1, cf. Fastl and Zwicker 2007, p. 160). This modification appears to be done in order to keep the dependencies simple and continuous, especially to keep CBWs constant at $100\,\mathrm{Hz}$ for frequencies below about $500\,\mathrm{Hz}$ that is for $n = 1, \ldots, 5$. Fastl and Zwicker (2007, p. 158) state:

> *"Although the lowest critical bandwidth in the audible frequency region may be very close to 80* Hz*, it is attractive to add the inaudible range from 0* Hz *to 20* Hz *to that critical band, and to assume that the lowest critical band ranges from 0* Hz *to 100* Hz*."*

Based on the updated data, Zwicker and Terhardt (1980) proposed the frequency dependent CBW function

$$\frac{\Delta f_{G_Z}(f)}{\mathrm{Hz}} = 25 + 75\left[1 + 1.4\left(\frac{f}{\mathrm{kHz}}\right)^2\right]^{0.69}, \quad (1)$$

fitting the updated data with an accuracy of ±10% (Fastl and Zwicker 2007, p. 164, dashed gray contour in figure 1), while deviating by 25% from the original value at $n = 1$.

Traunmüller (1990) derived a simplified set of analytic expressions for applying the critical-band concept to speech technology. He specified CBW as a function of the CBR z_T (cf. below) in the range $0.27\,\mathrm{kHz} < f < 5.8\,\mathrm{kHz}$ by

$$\frac{\Delta f_{G_T}(z_\mathrm{T}(f))}{\mathrm{Hz}} = \frac{52548}{\left(\frac{z_\mathrm{T}(f)}{\mathrm{Bark}}\right)^2 - \frac{52.56\, z_\mathrm{T}(f)}{\mathrm{Bark}} + 690.39}. \quad (2)$$

Equation 2 is shown for the full audible frequency range by the dotted black contour in figure 1, as a function valid for the whole audio spectrum is targeted here.

Critical Bandwidth: Proposal

Both previous formulae for the frequency dependence of CBW do *not* exhibit the desired properties, that is little deviation from the originally tabulated CBW data and $\Delta f(f) \leq 2f\ \forall f$. However, the function

$$\Delta f_{G_V}(f) = \Delta f_{G_Z}(f)\left(1 - \frac{1}{(38.73f/\mathrm{kHz})^2 + 1}\right), \quad (3)$$

with $0 \leq f \leq 20\,\mathrm{kHz}$

fulfills all requirements while fitting the sample values $\Delta f_G[n]$ tabulated by Zwicker (1961) with an accuracy of ±10% for $n = 1, \ldots, 24$.

The solid red contour in figure 2 shows CBW ac. to equation 3, the data tabulated by Zwicker (1961, dots), and $\Delta f_{G_Z}(f)$ proposed by Zwicker and Terhardt (1980, dashed gray). The dotted line indicates $\Delta f(f) = 2f$.

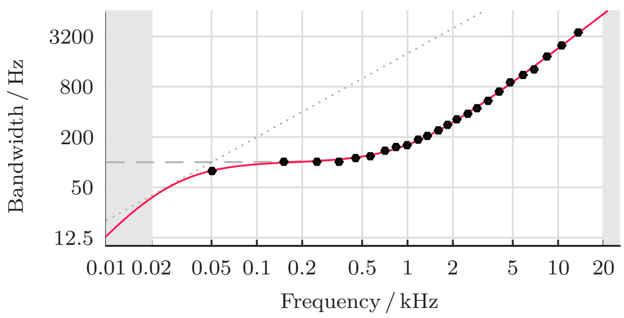

Figure 2: Critical bandwidth $\Delta f_G(f)$ ac. to the equation proposed here (solid red contour) and to Zwicker and Terhardt (1980, dashed gray). Black dots: tabulated original values ac. to Zwicker (1961); dotted gray line: $\Delta f(f) = 2f$.

Equivalent-Rectangular BW: Formulae

The ERB shown by the dotted black contour in figure 3 was originally defined by Moore and Glasberg (1983) by

$$\frac{\Delta f_{E_{M1}}(f)}{\mathrm{Hz}} = 6.23\left(\frac{f}{\mathrm{kHz}}\right)^2 + 93.39\,\frac{f}{\mathrm{kHz}} + 28.52. \quad (4)$$

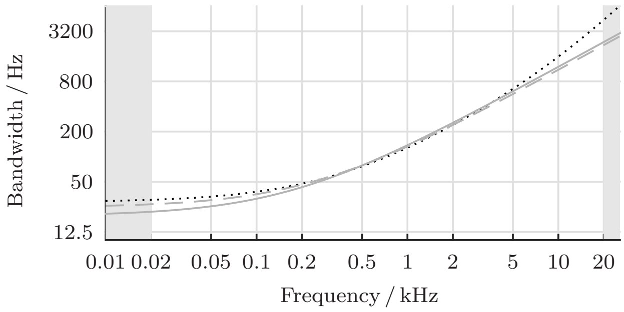

Figure 3: Equivalent-rectangular bandwidth $\Delta f_E(f)$ ac. to Moore and Glasberg (1983, dotted black contour; 1987 solid gray) as well as Glasberg and Moore (1990, dashed gray).

Two refinements to the original ERB formula have been given since then. The first,

$$\frac{\Delta f_{E_{M2}}(f)}{\mathrm{Hz}} = 19.5\left(6.046\frac{f}{\mathrm{kHz}} + 1\right) \quad (5)$$

proposed by Moore and Glasberg (1987), is indicated by the solid gray contour in figure 3. The most recent,

$$\frac{\Delta f_{E_M}(f)}{\mathrm{Hz}} = 24.7\left(4.37\frac{f}{\mathrm{kHz}} + 1\right), \quad (6)$$

is shown as a dashed gray curve in figure 3 and was specified by Glasberg and Moore (1990) using the same structure but slightly modified values (cf. Moore 2004, p. 73). Equation 6 represents the current standard method for computing the ERB at a given frequency.

None of the ERB formulae fulfills the desired low-frequency criterion $\Delta f(f) \leq 2f$. Instead, $\Delta f_{E_{M1}}(0) = 28.52\,\mathrm{Hz}$, $\Delta f_{E_{M2}}(0) = 19.5\,\mathrm{Hz}$, and $\Delta f_{E_M}(0) = 24.7\,\mathrm{Hz}$.

Equivalent-Rectangular BW: Proposal

Based on equation 6, the formula

$$\Delta f_{E_V}(f) = \Delta f_{E_M}(f)\left(1 - \frac{1}{(150f/\mathrm{kHz})^2 + 1}\right), \quad (7)$$

with $0 \leq f \leq 20\,\mathrm{kHz}$

is proposed for the ERB as a function of frequency, fulfilling $\Delta f(f) \leq 2f\ \forall f$. Within the audible frequency range, the largest deviation between $\Delta f_{E_V}(f)$ and $\Delta f_{E_M}(f)$ of about 10% occurs at $f = 20\,\mathrm{Hz}$. At $f = 70\,\mathrm{Hz}$, the deviation is less than 1% and stays below 1‰ for $f \geq 210\,\mathrm{Hz}$.

Figure 4 shows $\Delta f_{E_V}(f)$ as proposed here (solid red contour) and, for comparison purposes, $\Delta f_{E_M}(f)$ as given by Glasberg and Moore (1990, dashed gray).

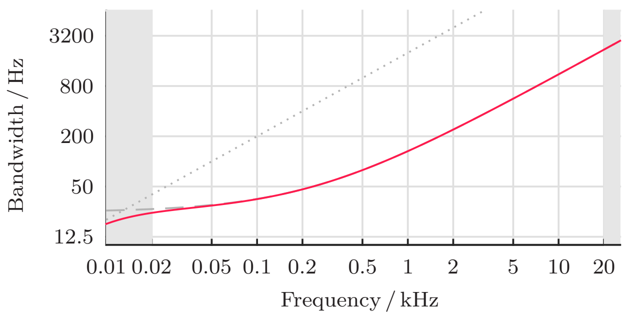

Figure 4: Equivalent-rectangular bandwidth $\Delta f_{\mathrm{E}}(f)$ as proposed here (solid red contour) and ac. to Glasberg and Moore (1990, dashed gray). Dotted gray line: $\Delta f(f) = 2f$.

Critical-Band Rate: Concept & Formulae

According to Fastl and Zwicker (2007, p. 158), the CBR was developed based on the finding that the human hearing system analyzes broadband sounds in spectral sections corresponding to the critical bands. Consequently, the frequency dependence of the CBR, assigned the unit Bark, is helpful for modeling characteristics of the human hearing system. The CBR function $z(f)$ is according to Fastl and Zwicker (2007, p. 159) defined by an interpolation of the integer valued sample points $z[m] = m\,\mathrm{Bark}$, with $m = 0, \ldots, 24$, corresponding to the frequencies $f_{\mathrm{l}}[m]$ (black dots in figure 5, Fastl and Zwicker 2007, p. 160).

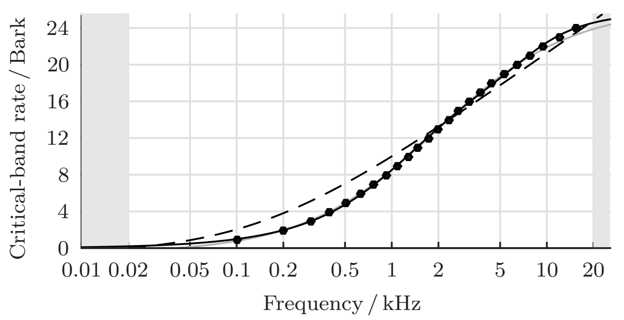

Figure 5: Critical-band rate $z(f)$ as a function of frequency f ac. to Zwicker and Terhardt (1980, black contour), Traunmüller (1990, gray), and Greenwood (1961, 1990, black dashed). Black dots: originally tabulated values ac. to Zwicker (1961).

The frequencies $f_{\mathrm{l}}[m]$ are given by the limits of 24 critical bands seamlessly arranged on the frequency scale, beginning at $f_{\mathrm{l}}[0] = 0$, so that the upper limiting frequency of each band with center frequency $f_{\mathrm{c}}[\kappa]$ equals the lower limiting frequency of the next higher band according to

$$f_{\mathrm{l}}[\kappa+1] = f_{\mathrm{l}}[\kappa] + \Delta f_{\mathrm{G_Z}}(f_{\mathrm{c}}[\kappa]), \quad \kappa = 0, \ldots, 23, \tag{8}$$

where $\Delta f_{\mathrm{G_Z}}(f_{\mathrm{c}}[\kappa])$ grows with κ that is with frequency (Fastl and Zwicker 2007, p. 160, black dots in figure 1). Zwicker and Terhardt (1980) proposed, based on the sample values given by Zwicker et al. (1957) and Zwicker (1961), the analytic expression

$$\frac{z_{\mathrm{Z}}(f)}{\mathrm{Bark}} = 13 \arctan \frac{0.76 f}{\mathrm{kHz}} + 3.5 \arctan \left(\frac{f}{7.5\,\mathrm{kHz}} \right)^2 \tag{9}$$

for the frequency dependence of CBR (solid black contour in figure 5, cf. Fastl and Zwicker 2007, p. 164). Equation 9 fits the sample points with an accuracy of ± 0.2 Bark. However, the applicability of equation 9 for the parameterization of auditory-adapted algorithms is limited, as it is not invertible in closed form. Furthermore, $z_{\mathrm{Z}}(f)$ has been proposed based on values tabulated for $0 \leq f \leq 15.5\,\mathrm{kHz}$, while nowadays hardware and algorithms often process signals with bandwidths exceeding the audible frequency range $20\,\mathrm{Hz} \leq f \leq 20\,\mathrm{kHz}$. Unfortunately, $z_{\mathrm{Z}}(f)$ tends to underestimate the CBR at $f > 16\,\mathrm{kHz}$. For example, $z_{\mathrm{Z}}(20\,\mathrm{kHz}) \approx 24.58$ Bark, while the CBW of the highest critical band tabulated is $\Delta f_{\mathrm{G_Z}}(f_{\mathrm{l}}[23]) = 3.5\,\mathrm{kHz}$. If the CBW is assumed to continue growing disproportionately with frequency for $f > 15.5\,\mathrm{kHz}$, a bandwidth in the range of $\Delta f_{\mathrm{G_Z}}(f_{\mathrm{l}}[24]) = 4.5\,\mathrm{kHz}$ appears to be a reasonable estimate. Hence, according to equation 8,

$$\begin{aligned} f_{\mathrm{l}}[25] &= f_{\mathrm{l}}[24] + \Delta f_{\mathrm{G_Z}}(f_{\mathrm{l}}[24]) \\ &= (15.5 + 4.5)\,\mathrm{kHz} = 20\,\mathrm{kHz} \end{aligned} \tag{10}$$

and the hypothetic CBR $z[25] = 25$ Bark would be reached at $f_{\mathrm{l}}[25] = 20\,\mathrm{kHz}$. Consequently, $z_{\mathrm{Z}}(20\,\mathrm{kHz}) \approx 24.58$ Bark according to equation 9 underestimates the CBR at $f = 20\,\mathrm{kHz}$.

The analytic expressions proposed by Traunmüller (1990) contain the invertible function

$$\frac{z_{\mathrm{T}}(f)}{\mathrm{Bark}} = 26.81 \frac{f/\mathrm{Hz}}{1960 + f/\mathrm{Hz}} - 0.53 \tag{11}$$

for the frequency dependence of CBR. This function is defined for $200\,\mathrm{Hz} < f < 6.7\,\mathrm{kHz}$, where it fits the original samples with an accuracy of ± 0.05 Bark, while deviating at frequencies outside this range by up to 0.73 Bark (gray contour in figure 5). Especially, $z_{\mathrm{T}}(0) \neq 0$, which is not in accordance with the definition of the CBR.

Assuming that positions on the basilar membrane correspond to CBRs, Greenwood (1961, 1990) derived relations between CBR and frequency from a cochlear frequency-position function based on physiological data of different species. Greenwood's frequency dependence of CBR

$$\frac{z_{\mathrm{G}}(f)}{\mathrm{Bark}} = 11.9 \log_{10} \left(\frac{f}{165.4\,\mathrm{Hz}} + 0.88 \right) \tag{12}$$

for humans (Greenwood 1990, dashed black contour in figure 5) is invertible, but deviates up to 2.17 Bark from the sample points originally tabulated by Zwicker (1961).

Critical-Band Rate: Proposal

None of the previously introduced formulae fulfills the requirements requested here. Therefore, the invertible relation of CBR to frequency

$$\frac{z_{\mathrm{V}}(f)}{\mathrm{Bark}} = 32.12 \left\{ 1 - \left[1 + \left(\frac{f/\mathrm{Hz}}{873.47} \right)^{1.18} \right]^{-0.4} \right\}, \tag{13}$$

with $0 \leq f \leq 20\,\mathrm{kHz}$

is proposed, approximating the values tabulated by Zwicker (1961) with an accuracy of $\pm 0.08\,\mathrm{Bark}$, in contrast to $\pm 0.2\,\mathrm{Bark}$ achieved by the noninvertible equation 9. Furthermore, equation 13 was fitted to the originally tabulated data extended by the pair $f_{\mathrm{l}}\,[25] = 20\,\mathrm{kHz}$ and $z\,[25] = 25\,\mathrm{Bark}$, so that more realistically $z_{\mathrm{V}}\,(20\,\mathrm{kHz}) \approx 24.86\,\mathrm{Bark}$. The inverse of equation 13,

$$\frac{f_{\mathrm{V}}\,(z)}{\mathrm{Hz}} = 873.47 \left[\left(\frac{32.12}{32.12 - z/\,\mathrm{Bark}} \right)^{2.5} - 1 \right]^{\frac{1}{1.18}}, \qquad (14)$$

$$\text{with } 0 \leq z \leq 24.86\,\mathrm{Bark},$$

allows for computing frequencies corresponding to a given distribution (e. g. equally spaced) on the CBR scale.

Equivalent-Rectangular Band Rate

Along with the ERB, Moore and Glasberg (1983) introduced the equivalent-rectangular band rate

$$\frac{z_{\mathrm{E}_1}\,(f)}{\mathrm{E}} = 11.17\,\log_e \left| \frac{f/\,\mathrm{kHz} + 0.312}{f/\,\mathrm{kHz} + 14.675} \right| + 43. \qquad (15)$$

The equivalent-rectangular band rate was given the unit E and is indicated by the gray contour in figure 6.

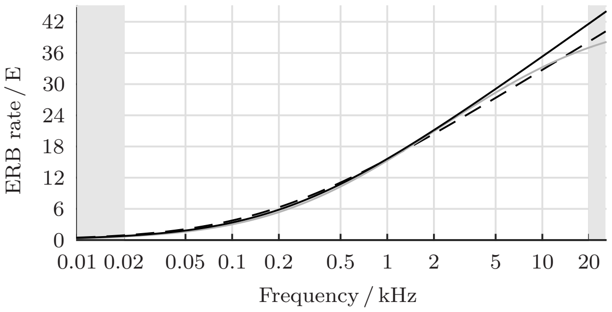

Figure 6: Equivalent-rectangular band rate $z_{\mathrm{E}}\,(f)$ ac. to Moore and Glasberg (1983, gray contour; 1987, dashed black) as well as Glasberg and Moore (1990, solid black).

Combined with the refinements of the ERB, the equivalent-rectangular band rate formula has been updated twice, first by Moore and Glasberg (1987) to

$$\frac{z_{\mathrm{E}_2}\,(f)}{\mathrm{E}} = 18.31\,\log_{10} \left(6.046 \frac{f}{\mathrm{kHz}} + 1 \right), \qquad (16)$$

(dashed black contour in figure 6), and then by Glasberg and Moore (1990) to

$$\frac{z_{\mathrm{E}}\,(f)}{\mathrm{E}} = 21.4\,\log_{10} \left(4.37 \frac{f}{\mathrm{kHz}} + 1 \right) \qquad (17)$$

(solid black contour in figure 6, cf. Moore 2004, p. 74). Both the latter formulae are invertible and can be implemented directly in signal processing algorithms.

Summary

Equations 3, 7, 13/14, and 17 provide a closed set of equations for applying the critical-band concept (including equivalent-rectangular bands) to digital signal processing. Artifacts and shortcomings as for example the undesired selection of negative frequencies or the non-invertibility of the critical-band rate function are avoided. Free software implementations of the proposed formulae are available from `http://www.windacoustics.com` (section `Downloads`).

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