Primary Source Correction (PSC) in Wave Field Synthesis

Florian Völk, Florian Lindner, Hugo Fastl

AG Technische Akustik, MMK, Technische Universität München, Germany, florian.voelk@mytum.de

Abstract

The theory of wave field synthesis is traditionally derived based on the Kirchhoff-Helmholtz integral equation for bounded volumes with continuous secondary point source distributions on the boundary area. Usually, a secondary source distribution on a boundary contour is used in practical implementations and the theory is adapted by reducing the geometry to two dimensions, employing secondary line sources perpendicular to the listening area. Consequently, the sound fields that can be synthesized are restricted to fields independent of the Cartesian coordinate direction defined by the line source axes, explicitly excluding the synthesis of spherical waves. Usually, secondary point sources are employed instead of line sources. In this case, correct synthesis is possible at one reference position in the listening area if secondary source correction is applied. In this paper, the theory of wave field synthesis is revised, resulting in a global formulation of secondary source correction that allows for correct amplitude and phase reproduction at the reference position. Based on that framework, primary source correction (PSC) is introduced. This procedure permits at the reference position correct synthesis of spherical waves evolving at arbitrary distances in the plane defined by the listening area, especially including the correct inverse proportionality of level and source distance, which is not possible using current approaches to the synthesis of spherical waves.

Introduction

Wave field synthesis (WFS) is an audio playback method, aiming at synthesizing the reference sound field (primary field) within a finite spatial region (Berkhout 1988). Theoretically, based on the Kirchhoff-Helmholtz integral equation (KHI), the primary field can be synthesized within a listening volume using an infinite number of secondary monopole and dipole point sources distributed continuously on the listening volume boundary (three-dimensional, 3D WFS, Vogel 1993). Typical primary fields are spherical waves (Boone et al. 1995). Current implementations attempt to reduce the number of secondary sources by degenerating the boundary area to a boundary contour (two-dimensional, 2D WFS, Spors et al. 2008). In this case, the situation is assumed to be independent of one arbitrary chosen Cartesian coordinate direction and the secondary point sources are replaced by secondary line sources with their axes along the coordinate direction selected (Spors and Ahrens 2010). The restriction to 2D situations prevents simulation of spherical wave fields, since no Cartesian coordinate direction can be found for a spherical wave to be independent of. This restriction is usually disregarded in the derivation of WFS, resulting in errors in the synthesized fields (Spors 2005, Spors et al. 2008). For typical implementations, the secondary line sources are replaced by point sources (2.5D WFS), requiring a correction term (Berkhout et al. 1993, Spors et al. 2008) and resulting in the fact that the synthesized field is correct at a reference position only for closed boundary contours (Start 1996) or on a reference line parallel to a linear secondary source distribution (de Vries 1996). All current secondary source corrections are based on far-field and high-frequency approximations, often disregarding phase relations, which results in erroneous synthetic fields, especially in the near-field and low-frequency regions (Start 1997, Spors et al. 2008).

In this paper, an updated and extended framework for WFS with continuous secondary source distributions is given, including a global formulation of secondary source correction. Furthermore, primary source correction (PSC) is introduced, which permits the correct synthesis of spherical waves evolving at arbitrary distances in the plane defined by the listening area and results in the correct inverse proportionality of level and distance for primary point sources at the correct absolute level, which is shown not to hold true for previous approaches of simulating primary spherical waves by means of WFS. The paper starts with a refinement of continuous 3D WFS in a listening volume, followed by the discussion of spherical and cylindrical primary sources. On that basis, the derivations of 2D and 2.5D WFS are revised, resulting in a general formulation of secondary source correction and the derivation of primary source correction (PSC), allowing for corrected reproduction of spherical waves. Finally, with regard to implementation, monopole only WFS is discussed. Time dependent variables are denoted by lower case, frequency dependent variables by upper case letters.

3D Wave Field Synthesis

Following Williams (1999), equation 8.15, the KHI can be written so that it describes the homogeneous acoustic pressure field $p(\mathbf{x})$ within the source-free volume V, with no field existing outside V (interior KHI). If $P(\mathbf{x})$ denotes the temporal Fourier transform of $p(\mathbf{x})$, the pressure field $p(\mathbf{x})$ is given by the KHI on basis of the sound pressure spectrum $P(\mathbf{x}_0)$ and its directional gradient

$$\frac{\partial}{\partial \mathbf{n}} P(\mathbf{x}_0) = \langle \nabla P(\mathbf{x}), \mathbf{n}(\mathbf{x}_0) \rangle \big|_{\mathbf{x}=\mathbf{x}_0} \tag{1}$$

(Bronstein et al. 2001, 13.34) on the surface with $\mathbf{x}_0 \in S_0$ and the inward normal $\mathbf{n}(\mathbf{x}_0)$ to the surface S_0 by

$$P(\mathbf{x}) = -\oiint_{S_0} \Big[G_{3\mathrm{D}}(\mathbf{x}|\mathbf{x}_0) \frac{\partial}{\partial \mathbf{n}} P(\mathbf{x}_0) - P(\mathbf{x}_0) \times \frac{\partial}{\partial \mathbf{n}} G_{3\mathrm{D}}(\mathbf{x}|\mathbf{x}_0) \Big] dS_0, \quad \forall\, \mathbf{x} \in V,\ \mathbf{x} \notin S_0,\ \mathbf{x}_0 \in S_0 \tag{2}$$

(the inward normal is typically used in WFS while the outward normal is employed by Williams 1999, therefore the sign is inverted in equation 2, cf. also Skudrzyk 1971, equation 20, p. 492). $G_{3D}(\mathbf{x}|\mathbf{x}_0)$ denotes the three-dimensional free space Green's function that can be regarded as representation of the spatio-temporal transfer characteristics of a monopole point source at $\mathbf{x}_0$ evaluated at the position $\mathbf{x} \neq \mathbf{x}_0$ (Spors et al. 2008, p. 2). According to Skudrzyk (1971) equation 29 on p. 645, the free space Green's function is given by

$$G_{3D}(\mathbf{x}|\mathbf{x}_0) = \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi\,|\mathbf{x}-\mathbf{x}_0|}, \qquad \forall\, \mathbf{x} \neq \mathbf{x}_0, \tag{3}$$

with the acoustical wave number

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \tag{4}$$

(Zollner and Zwicker 1993, with the frequency f, the speed of sound c and the wave length λ). The directional gradient of the free space 3D Green's function in boundary normal direction $\mathbf{n}(\mathbf{x}_0)$

$$\frac{\partial}{\partial \mathbf{n}} G_{3D}(\mathbf{x}|\mathbf{x}_0) = \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \mathrm{j}k\right) \frac{(\mathbf{x}-\mathbf{x}_0)^{\mathrm{T}}\mathbf{n}(\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|} \\ \times\; G_{3D}(\mathbf{x}|\mathbf{x}_0), \qquad \forall\, \mathbf{x} \neq \mathbf{x}_0 \tag{5}$$

(for the KHI to be taken with respect to $\mathbf{x}_0$, cf. Pierce 1998, p. 165 and 166) represents the spatio-temporal transfer characteristics of a dipole point source at $\mathbf{x}_0$ with its main axis in direction $\mathbf{n}(\mathbf{x}_0)$ and evaluated at $\mathbf{x} \neq \mathbf{x}_0$ (cf. Spors et al. 2008, p. 2 and 3).

In the context of WFS, the KHI can be interpreted in that the pressure field within the source free listening volume V can be controlled by an appropriately driven secondary source distribution on the volume surface S_0. The secondary source characteristics are described by the free space 3D Green's function (monopole point source) and its directional gradient (dipole point source), while the driving functions (the spectra of the signals the secondary sources are driven by) are given by the primary sound pressure spectrum and its directional gradient on S_0. Speaking descriptively, a continuous distribution of monopole and dipole secondary sound sources along S_0 allows to control the pressure field inside the source free volume V bounded by S_0 with no field outside V.

For synthesis of a specific primary field, each monopole secondary source is driven by the directional gradient of the primary pressure field at the respective secondary source position $\mathbf{x}_0$ along the inward normal $\mathbf{n}(\mathbf{x}_0)$ on S_0, and the dipole sources are driven by the primary pressure at $\mathbf{x}_0$. This situation is commonly referred to as 3D WFS (Vogel 1993, Spors et al. 2008). The listening volume can be degenerated to a half space with a boundary surface, the latter modeled as an infinitely extended plane and a half-sphere of infinite radius. The integration is then reduced to the plane, assuming that the field goes to zero at infinity (Spors et al. 2008, p. 7).

Primary Sources

In the temporal frequency domain, a primary sound field to be synthesized by WFS needs to be given by its sound pressure spectrum $P(\mathbf{x}_0)$ on the volume boundary S_0 with $\mathbf{x}_0 \in S_0$ and the corresponding directional gradient in direction of the inward normal $\mathbf{n}(\mathbf{x}_0)$ on S_0 (cf. equation 2). Consequently, arbitrary source free sound fields can be synthesized within V if the sound pressure distribution and its directional gradient on S_0 are known, either by measurement (data-based) or analytically, based on primary source models (model-based, cf. Vorländer 2008). Typically applied simple primary source models are plane, spherical, and cylindrical wave fields (Spors et al. 2008, p. 5). Assuming the sound pressure spectrum on a sphere with radius $a \to 0$ around a primary monopole point source according to Zollner and Zwicker (1993), equation 2.91 to be denoted by $\hat{P}_{\mathrm{a}}$, spherical and cylindrical primary source models are introduced in the following. More complex primary source models for WFS have been developed (Baalman 2007, Ahrens and Spors 2007), but are not within the scope of this study. The field variable is without loss of generality set to $\mathbf{x}_0 \in S_0$ here, and the models are given in form of the primary pressure field $P_{\mathrm{pf}}(\mathbf{x}_0)$, the corresponding directional gradient $\frac{\partial}{\partial \mathbf{n}} P_{\mathrm{pf}}(\mathbf{x}_0)$, and the local propagation direction $\mathbf{n}_{\mathrm{pf}}(\mathbf{x}_0)$ at $\mathbf{x}_0$.

Spherical waves

Spherical sound waves (SWs) are radiated concentrically from a point source. The sound pressure on a spherical shell is constant (Zollner and Zwicker 1993, section 2.3). Skudrzyk (1971) gives the spectrum

$$P_{\mathrm{sw}}(\mathbf{x}_0|\mathbf{x}_{\mathrm{p}}) = \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}_0-\mathbf{x}_{\mathrm{p}}|}}{|\mathbf{x}_0-\mathbf{x}_{\mathrm{p}}|}\hat{P}_{\mathrm{a}}, \quad \forall\, \mathbf{x}_0 \neq \mathbf{x}_{\mathrm{p}}, \tag{6}$$

of the pressure field created by a point source located at $\mathbf{x}_{\mathrm{p}}$ (equation 23 on p. 349). If the spherical wave is employed as primary source model for WFS, $\mathbf{x}_{\mathrm{p}} \notin V$ holds, for the KHI being valid for source free volumes. With the local propagation direction of the pressure field

$$\mathbf{n}_{\mathrm{sw}}(\mathbf{x}_0) = \frac{\mathbf{x}_0-\mathbf{x}_{\mathrm{p}}}{|\mathbf{x}_0-\mathbf{x}_{\mathrm{p}}|}, \quad \forall\, \mathbf{x}_0 \neq \mathbf{x}_{\mathrm{p}}, \tag{7}$$

the directional gradient of the field spectrum in direction $\mathbf{n}(\mathbf{x}_0)$ is given by

$$\frac{\partial}{\partial \mathbf{n}} P_{\mathrm{sw}}(\mathbf{x}_0|\mathbf{x}_{\mathrm{p}}) = \left(\frac{1}{|\mathbf{x}_0-\mathbf{x}_{\mathrm{p}}|} + \mathrm{j}k\right) \mathrm{e}^{\mathrm{j}\pi}\mathbf{n}_{\mathrm{sw}}^{\mathrm{T}}(\mathbf{x}_0) \\ \times\, \mathbf{n}(\mathbf{x}_0) P_{\mathrm{sw}}(\mathbf{x}_0|\mathbf{x}_{\mathrm{p}}), \qquad \forall\, \mathbf{x}_0 \neq \mathbf{x}_{\mathrm{p}}. \tag{8}$$

Cylindrical waves

Concentrically diverging cylindrical sound waves (CWs) can be regarded as if radiated from an infinitely long pulsating cylinder, a so-called line source. The resulting sound field is symmetric with regard to the cylinder axis, areas of constant pressure are cylinder shells (Zollner and Zwicker 1993, section 2.5), and it is independent of the Cartesian coordinate direction parallel to the cylinder axis. Here, the independence of one Cartesian coordinate direction is indicated by the upper index *2D*, for example $\mathbf{x}_0^{2\text{D}}$. According to Skudrzyk (1971), equation 34 and 36, p. 427, the sound pressure field spectrum of a line source located at $\mathbf{x}_\text{p}^{2\text{D}}$ is given by

$$P_\text{cw}(\mathbf{x}_0^{2\text{D}}|\mathbf{x}_\text{p}^{2\text{D}}) = \pi\, \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} H_0^{(2)}\left(k\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|\right)\hat{P}_\text{a}, \quad \forall\left(k\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|\right) > 0, \tag{9}$$

where $H_0^{(2)}\left(k\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|\right)$ denotes the Hankel function of the second kind, zeroth order. According to Abramowitz and Stegun (1972), equation 9.1.3 and 9.1.4, the Hankel functions $H_\text{n}^{(1)}(x)$ and $H_\text{n}^{(2)}(x)$ of the first and second kinds, nth order are for integer valued n defined by

$$H_\text{n}^{(1)}(x) = J_\text{n}(x) + \mathrm{j}Y_\text{n}(x) \qquad \text{and} \tag{10}$$

$$H_\text{n}^{(2)}(x) = J_\text{n}(x) - \mathrm{j}Y_\text{n}(x), \qquad \forall\, x > 0,\ n \in \mathbb{Z}_0^+, \tag{11}$$

with the Bessel functions of the first and second kinds $J_\text{n}(x)$ and $Y_\text{n}(x)$ (cf. Bronstein et al. 2001, p. 527). For large arguments, the approximation

$$H_\text{n}^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}}\, \mathrm{e}^{\mathrm{j}\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)} \qquad \text{and} \tag{12}$$

$$H_\text{n}^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}\, \mathrm{e}^{-\mathrm{j}\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)}, \quad \forall\, x \gg 1,\ n \in \mathbb{Z}_0^+ \tag{13}$$

(cf. Bronstein et al. 2001, equation 9.56a and 9.56c, p. 529). Taking into account the local propagation direction of a cylindrical pressure field

$$\mathbf{n}_\text{cw}(\mathbf{x}_0^{2\text{D}}) = \frac{\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}}{\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|}, \qquad \forall\, \mathbf{x}_0^{2\text{D}} \neq \mathbf{x}_\text{p}^{2\text{D}}, \tag{14}$$

the Hankel functions of the first and second kinds, zeroth order (equation 10 and 11) and

$$\frac{\partial}{\partial \mathbf{n}} H_0^{(2)}(k|\mathbf{x}|) = -\,k H_1^{(2)}(k|\mathbf{x}|)\,\frac{\mathbf{x}^\text{T}\mathbf{n}}{|\mathbf{x}|}, \quad \forall\,(k|\mathbf{x}|) > 0 \tag{15}$$

(cf. equation 1 in combination with Zollner and Zwicker 1993, equation 2.74), the spectrum of the sound pressure gradient in normal direction $\mathbf{n}(\mathbf{x}_0^{2\text{D}})$ at $\mathbf{x}_0^{2\text{D}}$ of a line source located at $\mathbf{x}_\text{p}^{2\text{D}}$ is given by

$$\frac{\partial}{\partial \mathbf{n}} P_\text{cw}(\mathbf{x}_0^{2\text{D}}|\mathbf{x}_\text{p}^{2\text{D}}) = k\pi\, \mathrm{e}^{\mathrm{j}\frac{\pi}{2}} H_1^{(2)}\left(k\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|\right) \times \mathbf{n}_\text{cw}^\text{T}(\mathbf{x}_0^{2\text{D}})\mathbf{n}(\mathbf{x}_0^{2\text{D}})\hat{P}_\text{a}, \quad \forall\left(k\left|\mathbf{x}_0^{2\text{D}} - \mathbf{x}_\text{p}^{2\text{D}}\right|\right) > 0. \tag{16}$$

2D Wave Field Synthesis

According to Williams (1999), p. 266, acoustical problems can be simplified assuming a three-dimensional field to be constant in one Cartesian coordinate direction. The resulting situation is referred to as two-dimensional WFS (Sonke et al. 1998, Sonke 2000, Spors et al. 2008), while the reproduction takes place in the 3D space. Following Williams (1999), a conversion from the 3D KHI to a 2D problem is possible neglecting the geometrical dependencies along the dimension the situation is assumed to be constant in and replacing the 3D by the 2D free space Green's function (Skudrzyk 1971, equation 87, p. 656)

$$G_\text{2D}(\mathbf{x}^{2\text{D}}|\mathbf{x}_0^{2\text{D}}) = \frac{\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}}{4} H_0^{(2)}\left(k\left|\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}}\right|\right), \quad \forall\left(k\left|\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}}\right|\right) > 0. \tag{17}$$

The 2D free space Green's function describes the spatio-temporal transfer characteristics of a monopole line source at $\mathbf{x}_0^{2\text{D}}$, extended in the constant coordinate direction, and evaluated at $\mathbf{x}^{2\text{D}}$ (cf. Spors et al. 2008). Using the Hankel function of the first kind, zeroth order and equation 15, the directional gradient of the 2D free space Green's function with respect to $\mathbf{x}_0^{2\text{D}}$ is given by

$$\frac{\partial}{\partial \mathbf{n}} G_\text{2D}(\mathbf{x}^{2\text{D}}|\mathbf{x}_0^{2\text{D}}) = k\frac{\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}}{4} H_1^{(2)}\left(k\left|\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}}\right|\right) \times \frac{(\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}})^\text{T}\mathbf{n}(\mathbf{x}_0^{2\text{D}})}{\left|\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}}\right|}, \quad \forall\left(k\left|\mathbf{x}^{2\text{D}} - \mathbf{x}_0^{2\text{D}}\right|\right) > 0. \tag{18}$$

The directional gradient of the 2D free space Green's function represents the spatio-temporal transfer characteristics of a dipole line source at $\mathbf{x}_0^{2\text{D}}$, extended in the constant coordinate direction, with its main axis in normal direction $\mathbf{n}(\mathbf{x}_0^{2\text{D}})$ and evaluated at $\mathbf{x}^{2\text{D}} \in (A \setminus x_0)$. With equation 17 and 18, the 3D KHI (equation 2) can be adapted to describe a two-dimensional situation: Within a source-free listening area A, the spectrum $P(\mathbf{x}^{2\text{D}})$ of the sound pressure field $p(\mathbf{x}^{2\text{D}}, t)$ is given by the spectrum of the pressure distribution on the area boundary contour x_0 and its directional gradient in inward normal direction $\mathbf{n}(\mathbf{x}_0^{2\text{D}})$ with $\mathbf{x}_0^{2\text{D}} \in x_0$ by

$$P(\mathbf{x}^{2\text{D}}) = -\oint_{x_0}\left[G_\text{2D}(\mathbf{x}^{2\text{D}}|\mathbf{x}_0^{2\text{D}})\frac{\partial}{\partial \mathbf{n}}P(\mathbf{x}_0^{2\text{D}}) - P(\mathbf{x}_0^{2\text{D}}) \times \frac{\partial}{\partial \mathbf{n}}G_\text{2D}(\mathbf{x}^{2\text{D}}|\mathbf{x}_0^{2\text{D}})\right]dx_0, \quad \forall\,\mathbf{x}^{2\text{D}} \in A,\ \mathbf{x}^{2\text{D}} \notin x_0,\ \mathbf{x}_0^{2\text{D}} \in x_0,\ k > 0. \tag{19}$$

In the context of WFS, equation 19 can be read in that the pressure field within the listening area A, bounded by x_0, and in all parallel areas can be controlled by an appropriately driven secondary line source distribution on x_0, with the line sources positioned perpendicular to A. The secondary source characteristics are given by the 2D free space Green's function (monopole line source) and its directional gradient (dipole line source). Here, appropriately driven means the monopole line sources driven by the directional gradient of the primary pressure field at the respective secondary source position in inward normal direction on x_0 and the dipole sources by the primary pressure at their positions. It is important to note that the primary field must not depend on the coordinate direction the secondary line sources (the 2D Green's functions) are independent of. This fact restricts primary fields that can be synthesized with 2D WFS to fields independent of one Cartesian coordinate direction, propagating perpendicular to the listening area, that is with their normal at every field point in the listening area. Components with other propagation directions *can not be synthesized correctly*, especially including primary spherical waves, regardless of their origin. The traditional derivation of WFS disregards this restriction (cf. Spors et al. 2008, equation 29), resulting in errors in the synthesized field.

Errors are here defined as ratio between the synthesized and the targeted field; a positive level error for example indicates the synthesized field to be of higher level than the reference field. Figure 1 shows the error of 2D WFS according to equation 19 of a primary spherical field (equations 6, 7, and 8) for a circular secondary source contour of 1.3 m radius, centered around the origin $\mathbf{x}_{\mathrm{a}} = [0\ 0\ 0]^{\mathrm{T}}$ of a Cartesian coordinate system.

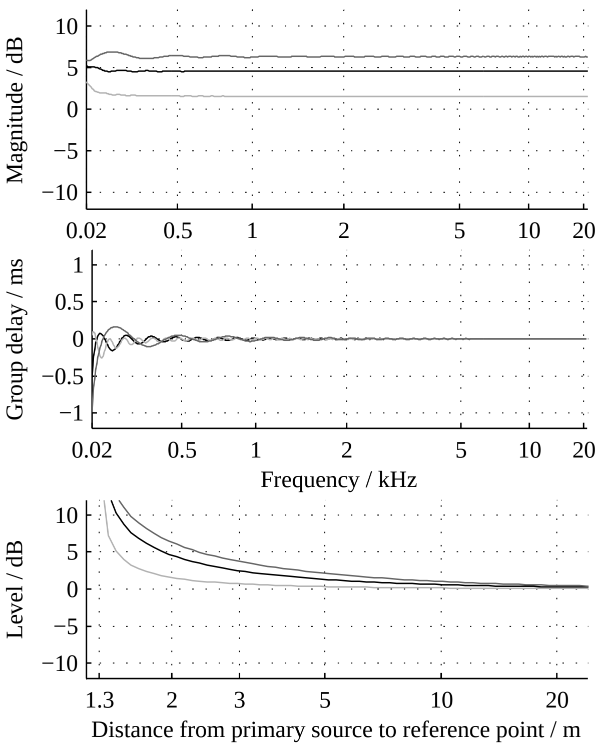

Figure 1: Deviation of the pressure field created by two-dimensional wave field synthesis (circular array at $\mathbf{x}_{\mathrm{a}} = [0\ 0\ 0]^{\mathrm{T}}$, 1.3 m radius) from the targeted spherical field at $\mathbf{x} = [0\ 0\ 0]^{\mathrm{T}}$ (black), $\mathbf{x} = [1\ 0\ 0]^{\mathrm{T}}$ (light gray), and $\mathbf{x} = [-1\ 0\ 0]^{\mathrm{T}}$ (dark gray). Magnitude and group delay error for fixed source at $\mathbf{x}_{\mathrm{p}} = [2\ 0\ 0]^{\mathrm{T}}$, magnitude error at $f = 2\,\mathrm{kHz}$ over the distance $|\mathbf{x} - \mathbf{x}_{\mathrm{p}}|$ between evaluation and source position.

Each panel holds data computed for three evaluation positions: the center of the secondary source contour $\mathbf{x} = [0\ 0\ 0]^{\mathrm{T}}$ (black), the position $\mathbf{x} = [1\ 0\ 0]^{\mathrm{T}}$ closer to the source (light gray) and the position $\mathbf{x} = [-1\ 0\ 0]^{\mathrm{T}}$ farther away from the source (dark gray). The evaluation positions lie on the line defined by the field origin and the secondary source contour midpoint and are selected so that the magnitudes and characteristics of all possible errors are reflected. Errors for evaluation positions shifted perpendicular to the line are comparable in their characteristics but of reduced magnitude. This geometrical configuration, especially the circular array and the evaluation positions are used throughout this paper. The magnitude errors are rather frequency independent at some 5 dB, varying by about ±3 dB for different evaluation positions within the secondary source distribution. Level deviations of more than 1 dB are audible (Fastl and Zwicker 2007, p. 180), and in the case of wave field synthesis most likely to result in erroneous distance perception, considering the absolute level to be an important cue for auditory distance perception (Zahorik et al. 2005). Group delay errors decrease with increasing frequency and reach values of some $\pm 100\,\mu$s at frequencies in the range around 50 Hz. Regarding temporal resolution, the human hearing system is most sensitive for interaural delays, an important cue for directional hearing with just noticeable interaural delays in the range of $50\,\mu$s (Fastl and Zwicker 2007, p. 293). This being said, the group delay errors at low frequencies are likely to be perceivable, possibly resulting in erroneous directional hearing. The lower panel shows exemplary the level error at $f = 2\,\mathrm{kHz}$ for different distances $|\mathbf{x} - \mathbf{x}_{\mathrm{p}}|$ between evaluation position $\mathbf{x}$ and primary source position $\mathbf{x}_{\mathrm{p}}$. For the magnitude error being rather frequency independent (cf. upper panel), the error at $f = 2\,\mathrm{kHz}$ represents a good indicator for the overall level error. The level error exceeds 10 dB for primary sources close to the secondary source contour, and decays globally with primary source distance. Consequently, distance perception in the synthetic field is likely to be influenced by the level error, especially for primary source distances below 10 m, where human distance perception is known to show the highest accuracy (Zahorik 2002, Völk 2010, Völk and Fastl 2011). Since the underlying problem is independent of the secondary source contour geometry, comparable errors result for circular secondary source contours of different radius and for differently shaped closed secondary source contours. Summarizing, the traditional procedure of simulating spherical waves in 2D WFS results in magnitude and group delay errors likely to be audible, with the level error dependent on the secondary source position.

2D Primary Source Correction (PSC)

The sound field of sources small compared to the wavelength can be approximated for large distances by a spherical wave (Zollner and Zwicker 1993, p. 75 to 76). This being said, the inability of 2D WFS to generate primary spherical waves means a severe restriction. For that reason, a procedure referred to as primary source correction (PSC) is introduced here, allowing for 2D WFS to synthesize the field of a primary point source located at $\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}$ in the plane defined by the listening area A but outside the listening area correctly at one reference position $\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}$ within the listening area. Using equations 9, 16, and 19, the pressure field of a primary cylindrical wave generated by 2D WFS originating at $\mathbf{x}_{\mathrm{p}}^{\mathrm{2D}} \notin (A \cup x_0)$ and with its normal in the listening area is described by

$$P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{2D}}) = -\oint\limits_{x_0} \left[G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}_0^{\mathrm{2D}}) \frac{\partial}{\partial \mathbf{n}} P_{\mathrm{cw}}(\mathbf{x}_0^{\mathrm{2D}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{2D}}) - \right.$$
$$\left. - P_{\mathrm{cw}}(\mathbf{x}_0^{\mathrm{2D}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{2D}}) \frac{\partial}{\partial \mathbf{n}} G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}_0^{\mathrm{2D}}) \right] dx_0,$$
$$\forall\, \mathbf{x}^{\mathrm{2D}} \in (A \setminus x_0)\,,\ \mathbf{x}_0^{\mathrm{2D}} \in x_0,\ \mathbf{x}_{\mathrm{p}}^{\mathrm{2D}} \notin (A \cup x_0)\,,\ k > 0. \tag{20}$$

It is possible without loss of generality to adjust the sound pressure in the field given in equation 20 at the reference position $\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} \in (A \setminus x_0)$ so that it represents the pressure in the field of a hypothetical primary point source positioned at $\mathbf{x}^{\mathrm{A}}_{\mathrm{p}} = \left(\mathbf{x}^{\mathrm{2D}}_{\mathrm{p}} \cap A\right)$ by

$$\begin{aligned} P_{\mathrm{sw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}) &= P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}) \frac{P_{\mathrm{sw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})}{P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})} \\ &= P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}) C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}). \end{aligned} \tag{21}$$

$C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})$ is referred to as the (complex-valued, frequency dependent) primary source correction factor

$$\begin{aligned} C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}) &= \frac{P_{\mathrm{sw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})}{P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})} \\ &= \frac{\mathrm{e}^{-\mathrm{j}\left(k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} - \mathbf{x}^{\mathrm{A}}_{\mathrm{p}}\right| - \frac{\pi}{2}\right)}}{\pi \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} - \mathbf{x}^{\mathrm{A}}_{\mathrm{p}}\right| H^{(2)}_0\left(k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} - \mathbf{x}^{\mathrm{A}}_{\mathrm{p}}\right|\right)}, \\ &\quad \forall \left(k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} - \mathbf{x}^{\mathrm{A}}_{\mathrm{p}}\right|\right) > 0, \end{aligned} \tag{22}$$

derived using equations 6 and 9. $C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}})$ is independent of the secondary source position $\mathbf{x}^{\mathrm{2D}}_0$ since PSC is done with respect to the primary source position $\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}$. The field of 2D WFS of a primary cylindrical wave with PSC is given (equations 20 and 21) by

$$\begin{aligned} P_{\mathrm{sw,psc,2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}) &= -\oint_{x_0} \Bigg[G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0) \\ \times \frac{\partial}{\partial \mathbf{n}} P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{2D}}_0|\mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}) &- P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{2D}}_0|\mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}) \frac{\partial}{\partial \mathbf{n}} G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0) \Bigg] \\ \times\, dx_0\, C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}), &\qquad \forall \mathbf{x}^{\mathrm{2D}}, \mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}} \in (A \setminus x_0), \\ \mathbf{x}^{\mathrm{2D}}_0 \in x_0,\ \mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}, &\mathbf{x}^{\mathrm{A}}_{\mathrm{p}} \notin (A \cup x_0),\ k > 0. \end{aligned} \tag{23}$$

The resulting field represents the field of a primary point source in the plane defined by the listening area A exclusively at the reference position $\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}$, while deviating from primary spherical and cylindrical wave fields at all other positions. Figure 2 shows the deviation between a spherical wave field originating at $\mathbf{x}_{\mathrm{p}} = [2\ 0\ 0]^{\mathrm{T}}$ approximated by 2D WFS with a primary cylindrical wave and PSC according to equation 23 and the targeted spherical field for the setup also shown in figure 1. The synthesis is correct at the reference position for all frequencies and primary source distances. The magnitude error at evaluation positions away from the reference position is approximately frequency independent and smaller than $\pm 3\,\mathrm{dB}$ (considering the evaluation positions selected representative for the error spectrum again), and of slightly higher magnitude in direction of the primary source, compared to directions farther away than the reference position. Group delay errors occur for evaluation away from the reference position at frequencies below $100\,\mathrm{Hz}$ in the range of $\pm 0.1\,\mathrm{ms}$. The lower panel indicates that the correct inverse proportionality of level and distance arises at the reference position, while deviations up to $5\,\mathrm{dB}$ are visible for distances below $10\,\mathrm{m}$ for evaluation positions farther and especially closer to the source than the reference position. The degeneration

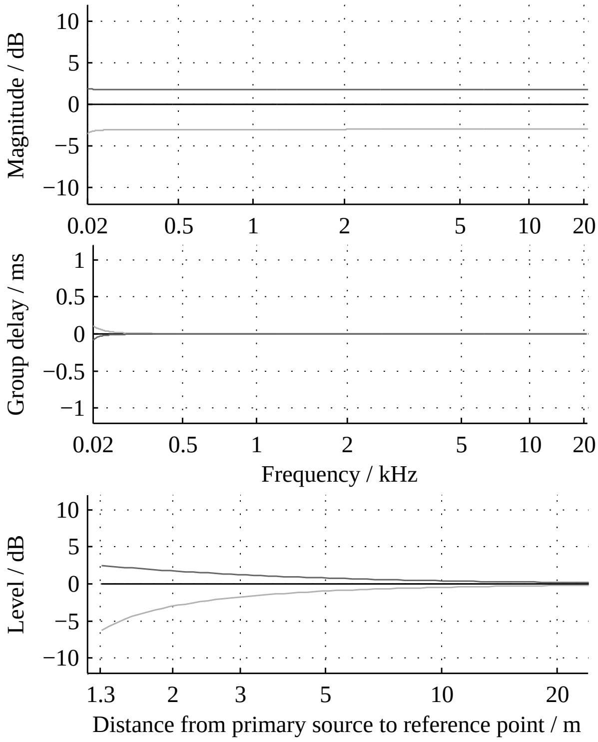

Figure 2: Deviation of the pressure field created by two-dimensional wave field synthesis from the targeted spherical field with a cylindrical primary field and primary source correction (PSC). Array geometry, primary source and evaluation positions according to figure 1.

of the area the synthesis is correct for to a reference position means a severe restriction of 2D WFS, but allows at the reference position for the synthesis of primary spherical wave fields originating in the plane defined by the listening area. This imposes no restriction compared to directly using primary spherical waves in 2D WFS, where the errors depend on the evaluation position, too (figure 1). Besides, as to be shown in the following, WFS is restricted to a reference position within the listening area if the secondary line are replaced by point sources, as done usually for WFS implementation.

2.5D Wave Field Synthesis

Further reduction in complexity is typically accomplished by replacing the secondary line sources in equation 19 by point sources located at $\mathbf{x}^{\mathrm{A}}_0$ in the listening area A and applying a correction term (cf. Spors et al. 2008, equation 24). $\mathbf{x}^{\mathrm{A}}_0 = (\mathbf{x}^{\mathrm{2D}}_0 \cap A)$ resembles the points where the secondary line sources intersect the listening area, that is $\mathbf{x}^{\mathrm{A}}_0 \in x_0$ holds. Due to the fact that in the considered situation 2D WFS is carried out with 3D secondary sources, this procedure is usually referred to as 2.5D WFS (cf. Spors et al. 2008). 2.5D WFS can be described based on equation 19 by replacing the 2D free space Green's functions by secondary source corrected 3D free space Green's functions, as well as the gradients of the 2D Green's functions by corrected 3D gradients. In the following, the (complex-valued, frequency dependent) correction factors $C_{\mathrm{ss},G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)$ and $C_{\mathrm{ss},\partial G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)$ are referred to as secondary source correction (SSC) factors, valid for the reference position $\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} \in (A \setminus x_0)$. Secondary source corrections have been proposed (Start 1997, Spors et al. 2008); here, a more general definition is given using equations 3 and 17 for secondary monopole sources by

$$\begin{aligned} C_{\mathrm{ss},G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) &= \frac{G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)}{G_{\mathrm{3D}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)} = \\ &= \pi \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right| H^{(2)}_0 \left(k \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|\right) \\ &\times \mathrm{e}^{\mathrm{j}\left(k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}-\mathbf{x}^{\mathrm{A}}_0\right|-\frac{\pi}{2}\right)}, \quad \forall \left(k \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|\right) > 0, \end{aligned} \tag{24}$$

and with 3, 5, and 18 for secondary dipole sources by

$$\begin{aligned} C_{\mathrm{ss},\partial G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) &= \frac{\frac{\partial}{\partial \mathbf{n}} G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)}{\frac{\partial}{\partial \mathbf{n}} G_{\mathrm{3D}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)} = \\ &= k\pi \frac{\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|^2}{1 + \mathrm{j}k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|} H^{(2)}_1 \left(k \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|\right) \\ &\times \mathrm{e}^{\mathrm{j}\left(k\left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}-\mathbf{x}^{\mathrm{A}}_0\right|-\frac{\pi}{2}\right)}, \quad \forall \left(k \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|\right) > 0. \end{aligned} \tag{25}$$

The wave field generated by 2.5D WFS can be computed by replacing $G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0)$ by $G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0)\, C_{\mathrm{ss},G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)$ and $\frac{\partial}{\partial \mathbf{n}} G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0)$ by $\frac{\partial}{\partial \mathbf{n}} G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0)\, C_{\mathrm{ss},\partial G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0)$ in equation 19, using equation 24 and 25. A 3D sound field arises, given by

$$\begin{aligned} P_{\mathrm{2.5D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}) &= -\oint_{x_0} \Big[G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0)\, C_{\mathrm{ss},G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) \\ &\times \frac{\partial}{\partial \mathbf{n}} P(\mathbf{x}^{\mathrm{2D}}_0) - P(\mathbf{x}^{\mathrm{2D}}_0) \frac{\partial}{\partial \mathbf{n}} G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0) \\ &\times C_{\mathrm{ss},\partial G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) \Big] dx_0, \quad \forall \mathbf{x} \in (A \setminus x_0)\,,\ \mathbf{x}^{\mathrm{A}}_0 \in x_0, \\ &\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} \in (A \setminus x_0)\,,\ \mathbf{x}^{\mathrm{A}}_0 = (\mathbf{x}^{\mathrm{2D}}_0 \cap A)\,,\ k > 0. \end{aligned} \tag{26}$$

The sound field described by equation 26 represents the intended field exclusively at the reference position. In general, no Cartesian coordinate system exists, where the resulting field is independent of one coordinate direction.

2.5D Primary Source Correction (PSC)

Primary source correction is directly applicable to 2.5D WFS because of its independence of the secondary sources and their positions. The wave field resulting from 2.5D WFS of a primary cylindrical wave with PSC is given using equations 9, 16, 22, and 26 by

$$\begin{aligned} &P_{\mathrm{sw,psc,2.5D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}) = -\oint_{x_0} \Big[G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0) \\ &\times C_{\mathrm{ss},G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) \frac{\partial}{\partial \mathbf{n}} P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{2D}}_0|\mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}) - P_{\mathrm{cw}}(\mathbf{x}^{\mathrm{2D}}_0|\mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}) \\ &\times \frac{\partial}{\partial \mathbf{n}} G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0)\, C_{\mathrm{ss},\partial G}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}|\mathbf{x}^{\mathrm{A}}_0) \Big] dx_0\, C_{\mathrm{ps}}(\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}|\mathbf{x}^{\mathrm{A}}_{\mathrm{p}}), \\ &\forall \mathbf{x} \in (A \setminus x_0)\,,\ \mathbf{x}^{\mathrm{A}}_{\mathrm{ref,psc}}, \mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} \in (A \setminus x_0)\,,\ \mathbf{x}^{\mathrm{A}}_0 \in x_0, \\ &\mathbf{x}^{\mathrm{A}}_0 = (\mathbf{x}^{\mathrm{2D}}_0 \cap A)\,,\ \mathbf{x}^{\mathrm{2D}}_{\mathrm{p}}, \mathbf{x}^{\mathrm{A}}_{\mathrm{p}} \notin (A \cup x_0)\,,\ k > 0. \end{aligned} \tag{27}$$

Equation 27 simplifies if identical reference positions are selected for primary and secondary source correction.

Secondary Monopole Sources Only

It has been shown by Copley (1968) that it is possible to achieve control over the sound pressure field within a listening volume using a secondary monopole distribution on the listening volume surface S_0. This procedure is referred to as *simple source formulation* (cf. also Williams 1999). Another approach to discard one of the contributions to the KHI is to use Green's functions that either equal zero on S_0 or that show zero gradients on S_0 (Dirichlet or Neumann Green's functions, cf. Williams 1999). In WFS, typically an approximative approach is employed to reduce the situation to the monopole only case (cf. Spors et al. 2008): it can be shown that

$$G_{\mathrm{N,p}}(\mathbf{x}|\mathbf{x}_0) = 2G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}_0) \qquad \text{and} \tag{28}$$

$$G_{\mathrm{N,l}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0) = 2G_{\mathrm{2D}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0) \tag{29}$$

represent Neumann Green's functions for planar respectively linear boundaries S_0 or x_0 (cf. Williams 1999, section 8.8.3). This holds not true for arbitrarily shaped boundaries. However, the artifacts resulting from applying $G_{\mathrm{N,p}}(\mathbf{x}|\mathbf{x}_0)$ or $G_{\mathrm{N,l}}(\mathbf{x}^{\mathrm{2D}}|\mathbf{x}^{\mathrm{2D}}_0)$ and the assumption of Neumann boundary conditions on S_0 or x_0 for arbitrarily shaped boundaries can be reduced by deactivating secondary sources at whose positions the local propagation direction $\mathbf{n}_{\mathrm{pf}}(\mathbf{x}_0)$ of the primary field (cf. section *Primary Sources*) features no positive component in direction of the boundary normal $\mathbf{n}(\mathbf{x}_0)$ (Spors 2007a). This is carried out by multiplication of the driving functions by the so-called secondary source activation factor

$$a(\mathbf{x}_0) = \begin{cases} 1 & \text{if } \langle \mathbf{n}_{\mathrm{pf}}(\mathbf{x}_0), \mathbf{n}(\mathbf{x}_0)\rangle > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{30}$$

defined here in principle according to Spors (2007a,b), but based on geometrical criteria rather than the primary field intensity vector. This procedure results in general in two major consequences: a sound field outside the listening area or listening volume arises and the field created inside the listening area or volume deviates from the primary field (cf. Spors et al. 2008). To ensure that no contributions of the erroneous outer field propagate into the listening area or volume and interfere with the field created intentionally, convex listening area or volume boundaries are required. In the following, monopole only 2.5D WFS is discussed, including the derivation of so-called driving functions $D(\mathbf{x}_0)$ allowing for the synthesis of spherical primary sound fields. Driving functions represent the spectra of the signals the secondary monopole sources have to be driven with to generate the target primary sound field. Equation 26 can be adapted to resemble 2.5D monopole WFS with the reference point $\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}$ in the listening area A by

$$\begin{aligned} P_{\mathrm{2.5D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}) &= \oint_{x_0} D_{\mathrm{2.5D}}(\mathbf{x}^{\mathrm{A}}_0|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}}) \\ &\times G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}^{\mathrm{A}}_0) dx_0, \forall\, \mathbf{x} \neq \mathbf{x}^{\mathrm{A}}_0, \left(k \left|\mathbf{x}^{\mathrm{A}}_{\mathrm{ref,ssc}} - \mathbf{x}^{\mathrm{A}}_0\right|\right) > 0, \end{aligned} \tag{31}$$

with the driving functions for 2.5D monopole WFS

$$
\begin{aligned}
&D_{\mathrm{2.5D}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}) = \\
&\quad = -2a(\mathbf{x}_0^{\mathrm{A}})\, C_{\mathrm{ss},G}(\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_0^{\mathrm{A}})\, \frac{\partial}{\partial \mathbf{n}} P(\mathbf{x}_0^{\mathrm{2D}}).
\end{aligned} \tag{32}
$$

Derived from 2D WFS, 2.5D monopole WFS is also not capable of generating spherical waves at arbitrary positions (cf. section *2.5D Wave Field Synthesis*). The 2.5D scenario with PSC with respect to $\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}$ for the reproduction of a primary spherical wave generated by a point source at $\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}$ within the listening area A is derived from equation 27 to

$$
\begin{aligned}
&P_{\mathrm{sw,psc,2.5D}}(\mathbf{x}|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}) = \\
&\quad = \oint_{x_0} D_{\mathrm{2.5D,psc}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}) \\
&\qquad \times G_{\mathrm{3D}}(\mathbf{x}|\mathbf{x}_0^{\mathrm{A}})dx_0, \qquad \forall\, \mathbf{x} \neq \mathbf{x}_0^{\mathrm{A}}, \\
&\left(k\left|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|\right) > 0, \left(k\left|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|\right) > 0.
\end{aligned} \tag{33}
$$

The primary and secondary source corrected 2.5D driving functions are then given by

$$
\begin{aligned}
&D_{\mathrm{2.5D,psc}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}) = \\
&\quad = -2a(\mathbf{x}_0^{\mathrm{A}})\, C_{\mathrm{ss},G}(\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_0^{\mathrm{A}})\, C_{\mathrm{ps}}(\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}) \\
&\qquad \times \frac{\partial}{\partial \mathbf{n}} P_{\mathrm{cw}}(\mathbf{x}_0^{\mathrm{2D}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{2D}}).
\end{aligned} \tag{34}
$$

Comparison to Previous Approaches

Previously derived driving functions for 2.5D monopole WFS are based on the large-argument approximation of the Hankel functions (cf. equations 12 and 13) for secondary source correction, and therefore known to produce synthesis errors for evaluation positions close to the secondary source contours and at low frequencies (cf. Spors and Ahrens 2010), since the approximations are valid only for large arguments. The comparisons are given here implementing the primary and secondary source corrections proposed in equation 23 and 24 for the approaches introduced in this paper using the Hankel functions (equation 10 and 11), but can be done also with their large-argument approximations (equation 12 and 13).

The early formulations of WFS are given specifically for linear or curved but infinitely extended secondary source contours (cf. Vogel 1993, Start 1996, Verheijen 1997). As a consequence, numerical simulations or implementations need to limit the integration region, what introduces artifacts in the generated field, which are typically reduced by applying a spatial truncation window (Start 1997, section 4). Since methods for truncation are beyond the scope of this paper, the driving functions given by Verheijen (1997) are included in the comparison for 2.5D WFS, but applied to closed secondary source contours without truncation, using the secondary source activation factor given in equation 30. Expressing the cosine of the angle between the secondary source contour and the vector $\left(\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right)$ by the scalar product of the vector and the contour normal, equation 2.32b in Verheijen (1997) can be given using equation 34 in Spors et al. (2008) and the nomenclature introduced here by

$$
\begin{aligned}
D_{\mathrm{2.5D,sw,ve}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref}}^{\mathrm{A}}) &= a(\mathbf{x}_0^{\mathrm{A}}) \sqrt{\frac{k}{2\pi}}\, \frac{\mathrm{e}^{-\mathrm{j}\left(k\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right| - \frac{\pi}{4}\right)}}{\sqrt{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|}} \\
&\times \sqrt{\frac{\left|\mathbf{x}_{\mathrm{ref}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|}{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right| + \left|\mathbf{x}_{\mathrm{ref}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|}}\, \frac{(\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}})^{\mathrm{T}}\mathbf{n}(\mathbf{x}_0^{\mathrm{A}})}{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|}\, \hat{P}_{\mathrm{a}}.
\end{aligned} \tag{35}
$$

Figure 3 shows the error of the synthesis of spherical primary fields by 2.5D WFS using the driving functions given in equation 35 for the array geometry and evaluation positions according to figure 1. The magnitude and group delay errors are approximately frequency independent and vanish at the optimization position for frequencies above about 500 Hz, apart from a global offset of about -22 dB, corresponding to the factor $1/(4\pi)$. At lower frequencies, errors and frequency dependencies occur. At the optimization position, no primary source position dependent level error occurs for distances larger than about 1.5 m. Away from the optimization position, distance dependent errors are visible.

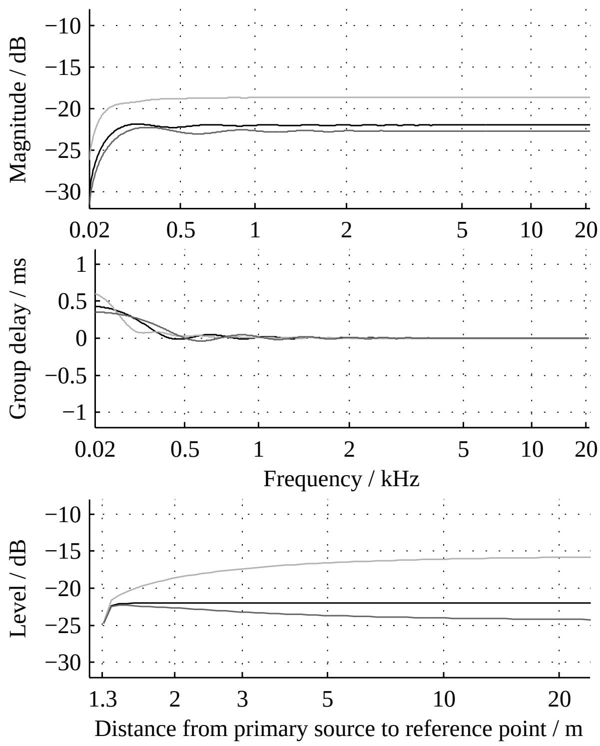

Figure 3: Deviation of the pressure field created by 2.5D wave field synthesis with modifications according to Verheijen (1997) from the targeted spherical field. Array geometry, primary source and evaluation positions according to figure 1.

Equation 29 given by Spors et al. (2008) resembles another driving function for the generation of spherical primary fields by 2.5D WFS, derived introducing a primary spherical field in the 2D KHI. According to section *2D Wave Field Synthesis*, this procedure is expected to result in errors in the generated field. Using the nomenclature introduced here, Spors' driving functions are given by

$$D_{\mathrm{2.5D,sw,sp}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref}}^{\mathrm{A}}) = 2a(\mathbf{x}_0^{\mathrm{A}}) \left(\frac{1}{\mathrm{j}k\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|} + 1 \right) \hat{P}_{\mathrm{a}} \times \sqrt{\frac{2\pi k \left|\mathbf{x}_{\mathrm{ref}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|}{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|}} \frac{\mathrm{e}^{-\mathrm{j}\left(k\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right| - \frac{\pi}{4}\right)}}{\sqrt{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|}} \frac{(\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}})^{\mathrm{T}} \mathbf{n}(\mathbf{x}_0^{\mathrm{A}})}{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|}. \tag{36}$$

Figure 4 shows the errors of 2.5D WFS of spherical primary fields using equation 36.

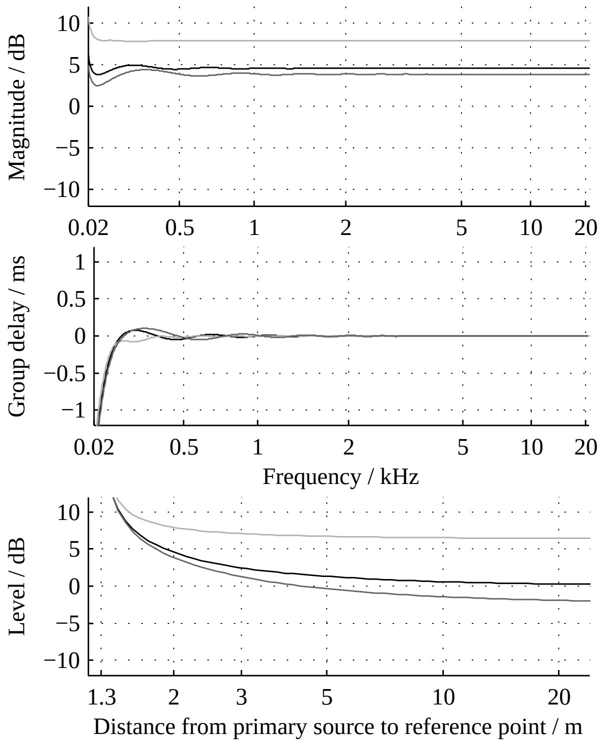

Figure 4: Deviation of the pressure field created by 2.5D wave field synthesis according to Spors et al. (2008) from the targeted spherical field. Array geometry, primary source and evaluation positions according to figure 1.

At all evaluation positions, a frequency independent magnitude error occurs, in addition to low frequency deviations. The level errors at all evaluation positions depend on the distance to the primary source.

With equations 14, 16, 22, 24, and 34, the 2.5D WFS of a primary spherical wave based on the framework introduced here (that is synthesizing a primary cylindrical source with PSC) is given by

$$\begin{aligned} D_{\mathrm{2.5D,psc}}(\mathbf{x}_0^{\mathrm{A}}|\mathbf{x}_{\mathrm{p}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}}|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}}) &= 2a(\mathbf{x}_0^{\mathrm{A}}) \left|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right| \\ &\times \frac{\mathrm{e}^{-\mathrm{j}\left(k\left|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right| - k\left|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right| + \frac{\pi}{2}\right)}}{\left|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|} \frac{(\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}})^{\mathrm{T}} \mathbf{n}(\mathbf{x}_0^{\mathrm{A}})}{\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|} \\ &\times k\pi \hat{P}_{\mathrm{a}} H_1^{(2)}\left(k\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|\right) \frac{H_0^{(2)}\left(k\left|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|\right)}{H_0^{(2)}\left(k\left|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|\right)}, \\ \forall\ \left(k\left|\mathbf{x}_{\mathrm{ref,psc}}^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|\right) > 0, &\left(k\left|\mathbf{x}_{\mathrm{ref,ssc}}^{\mathrm{A}} - \mathbf{x}_0^{\mathrm{A}}\right|\right) > 0, \left(k\left|\mathbf{x}_0^{\mathrm{A}} - \mathbf{x}_{\mathrm{p}}^{\mathrm{A}}\right|\right) > 0. \end{aligned} \tag{37}$$

In figure 5, the errors resulting for 2.5D wave field synthesis of a spherical primary field using the method derived here are depicted.

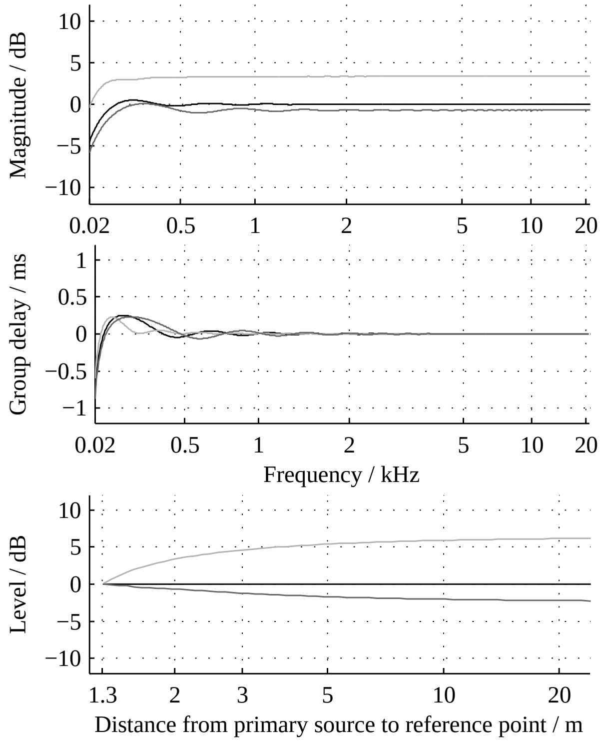

Figure 5: Deviation of the pressure field created by 2.5D wave field synthesis from a spherical field using a cylindrical primary field and primary source correction (PSC). Geometry, primary source and evaluation positions according to figure 1.

At the optimization position, magnitude and group delay errors occur below approximately 500 Hz due to the approximated Neumann Green's function (cf. equation 29). The level at 2 kHz is correct at the optimization position for all distances to the primary source. Away from the optimization point, distance dependent level errors occur, which are larger for positions in source direction.

Conclusions

In this paper, the basic theory of wave field synthesis is reconsidered, resulting in a global formulation of the secondary source correction that reduces the synthesis error over the whole listening area and allows for the correct amplitude and phase reproduction at the (secondary source correction) reference position.

Furthermore, primary source correction (PSC) is introduced, which permits at a reference position the correct synthesis of spherical waves evolving in the plane defined by the listening area, including the correct inverse proportionality of level and distance.

Acknowledgments

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